9/10

Section 12.5: Lines & Planes

A line in 3-space is given by parametized vector equation $\boxed{\ell(t) = \vec{p} + t\vec{v}}$ where $\vec{p}$ = position vector of a point on ℓ and $\vec{v}$ = direction of line

Ex: Compute the vector equation of the line through $(-6, 2, 3)$ and parallel to line
$m(t) = \langle 0, 2, -1 \rangle + t\langle -2, 1, 5 \rangle$

sol

Given: $\vec{p} = \langle -6, 2, 3 \rangle$. Because of parallelism, $\vec{v} = \langle -2, 1, 5 \rangle$ is a valid direction vector.

$\therefore \ \ell(t) = \langle -6, 2, 3 \rangle + t\langle -2, 1, 5 \rangle$

The parametric equations of a line are $\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$ which are the component functions of the vector form

Ex: For l and m as in the previous example, simplify the vector equation:

$l(t) = \langle -6-2t, 2+t, 3+5t \rangle$ $\quad \begin{cases} x = -6-2t \\ y = 2+t \\ z = 3+5t \end{cases}$

$m(t) = \langle -2t, 2+t, -1+5t \rangle$ $\quad \begin{cases} x = -2t \\ y = 2+t \\ z = -1+5t \end{cases}$

$\therefore$ has parametric equations

a line can also be represented (sometimes) by symetric equations

(solved for parameter) $\rightsquigarrow$ $\boxed{\dfrac{x-x_0}{a} = \dfrac{y-y_0}{b} = \dfrac{z-z_0}{c}}$

Ex: For l as above, has parametric equations:

$\begin{cases} x = -6-2t \\ y = 2+t \\ z = 3+5t \end{cases} \rightsquigarrow \begin{cases} \dfrac{x+6}{-2} = t \\ \dfrac{y-2}{1} = t \\ \dfrac{z-3}{5} = t \end{cases} \rightsquigarrow \boxed{\dfrac{x+6}{-2} = \dfrac{y-2}{1} = \dfrac{z-3}{5}}$

Symetric equations of l

This procedure can be done in reverse too; if given symetry equations we can easily get parametric equations too

Some Terminology: Two lines are ...

1. parallel if their direction vectors are parallel
2. intersecting if they have a point in common
3. skew if they are neither parallel or intersecting

ex: Classify as parallel, intersecting, or skew:

$$l_1(t) = \langle 5-12t, 3+9t, 1-3t \rangle$$
$$l_2(t) = \langle 3+8t, -6t, 7+2t \rangle$$

sol: $l_1(t) = \langle 5,3,1 \rangle + t\langle -12,9,-3 \rangle$
$l_2(t) = \langle 3,0,7 \rangle + t\langle 8,-6,2 \rangle$] not equal, but we must check the unit vectors

unit vectors:

$$\frac{1}{|\vec{v_1}|}\vec{v_1} = \frac{1}{\sqrt{234}}\langle -12,9,-3 \rangle = \frac{1}{3\sqrt{26}}\langle -12,9,-3 \rangle = \boxed{\frac{1}{\sqrt{26}}\langle -4,3,-1 \rangle}$$

$$\frac{1}{|\vec{v_2}|}\vec{v_2} = \frac{1}{\sqrt{109}}\langle 8,-6,2 \rangle = \frac{1}{2\sqrt{26}}\langle 8,-6,2 \rangle = \boxed{\frac{1}{26}\langle 4,-3,1 \rangle}$$

$$\boxed{\text{Notice: } \frac{1}{|\vec{v_2}|}\vec{v_2} = -\frac{1}{|\vec{v_1}|}\vec{v_1}, \text{ so } l_1 \text{ is parallel to } l_2}$$

check if they're intersecting:
Do NOT solve $l_1(t_0) = l_2(t_0)$. Instead, solve for something like $l_1(t) = l_2(s) \Rightarrow$ do paths cross, not necessarily at the same time:

$$\langle 5-12t, 3+9t, 1-3t \rangle = \langle 3+8s, -6s, 7+2s \rangle$$

i.e. $\begin{cases} 5-12t = 3+8s \\ 3+9t = -6s \\ 1-3t = 7+2s \end{cases} \Rightarrow \begin{cases} -12t-8s = -2 \\ 9t+6s = -3 \\ -3t-2s = 6 \end{cases} \Rightarrow$ This is a system of equations

Ex cont: $\begin{cases} 6t+4s=1 \end{cases} \Rightarrow \begin{cases} 3t+2s=-1 \\ 3t+2s=-6 \end{cases}$ implies $-1 = 3t+2s = -6$ which is simply not a valid expression.

$\therefore$ the lines are not intersecting

Recall: A plane in 3-space has vector equation

$$\vec{n} \cdot (\vec{x} - \vec{p}) = 0$$

normal vector — vector of variables — any position point on the plane

Ex: Compute the plane through $(1,2,4)$ and perpendicular to $\langle -2,1,3\rangle$:

$$\vec{n}\cdot(\vec{x}-\vec{p}) = 0 \Rightarrow \langle -2,1,3\rangle \cdot \langle x-1, y-2, z-4\rangle$$

$$= -2(x-1) + 1(y-2) + 3(z-4) = 0$$

Ex: Compute the plane through the point $(3,5,-1)$ and containing the line $\begin{cases} x = 4-t \\ y = 2t-1 \\ z = -3t \end{cases}$

Sol: $\vec{p} = \langle 3,5,-1\rangle$
need Q, a point on $l \Rightarrow$ pick a time
so lets use $Q = l(0) = (4,-1,0)$

So, $\vec{u} = \langle 3-4, 5--1, -1-0\rangle = \langle -1,6,-1\rangle$

$l(t) = \langle 4,-1,0\rangle + t\underbrace{\langle -1,2,-3\rangle}_{\vec{v}}$

Ex Cont: $\vec{n} = \vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 6 & -1 \\ -1 & 2 & -3 \end{vmatrix} = \langle -16, -2, 4 \rangle$

$\therefore$ the plane has equation $\vec{n} \cdot (\vec{x} - \vec{p}) = 0$

i.e. $\langle -16, -2, 4 \rangle \cdot \langle x-3, y-5, z+1 \rangle = 0$
$\Rightarrow -16(x-3) - 2(y-5) + 4(z+1) = 0$

Section 12.6: Quadratic Surfaces

IDEA: We want to study degree 2 polynomials and solution sets in 3-space

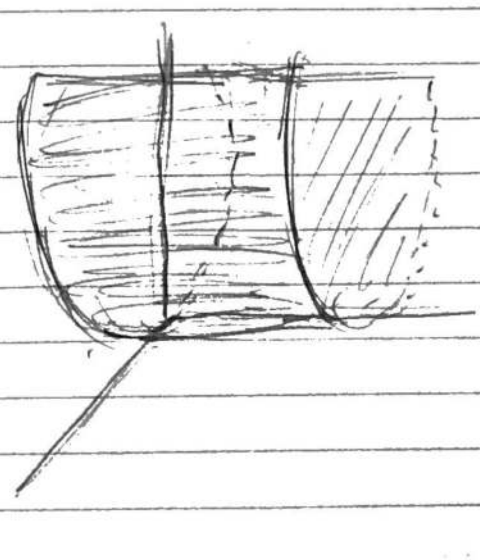
Ex: $p(x,y,z) = x^2 - z$ ← "degenerate" b/c it doesn't depend on ALL variables

solution set: $p(x,y,z) = 0$ iff $x^2 - z = 0$
$\Rightarrow x^2 = z$

In the xz plane, this looks like a parabola

This solution set is actually a (parabolic) cylinder

A picture in 3-space:



Kinda like a hot dog bun